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Certain Identities in K^h- Generalized Birecurrent Finsler Space

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Abstract: **We presented a Finsler space F_n whose Cartan's fourth curvature tensor K^i_{jkh} satisfies $K^i_{jkh|\ell\,|m} = \lambda_\ell\, K^i_{jkh|m} + b_{\ell m}\, K^i_{jkh}$, $K^i_{jkh} \neq 0$, where λ_ℓ and $b_{\ell m}$ are non-zero covariant vector field and covariant tensor field of second order, respectively. such space is called as K^h*–generalized birecurrent Finsler space* and denoted briefly by $K^h\!-GBR\!-F_n$, In the present paper we introduce some certain identities satisfies the generalized birecurrence property in our space.**

Keywords: **Finsler space, K^h– Generalized birecurrent Finsler space, Ricci tensor.**

1. INTRODUCTION

N. S. H. Hussien [4] obtained certain identities in a $K^h - recurrent$, M.A. A. Ali [1] obtained certain identities in a K^h– birecurrent Finsler space.

Let F_n be An n-dimensional Finsler space equipped with the metric function a $F(x,y)$ satisfying the request conditions [7].

The vectors y_i , y^i and the metric tensor g_{ij} satisfies the following relations

(1.1) $a)\ \ y_{i|k} = 0 \qquad b)\ \ y^i_{\ |k} = 0 \qquad c)\ \ g_{ij|k} = 0 \quad and \quad d)\ \ g^{ij}_{\ |k} = 0.$

The tensor C_{ijk} is known as $(h)hv$ - torsion tensor [5], it is positively homogeneous of degree -1 in y^i and symmetric in all its indices. By using Euler's theorem on homogeneous properties, this tensor satisfies the following identities

(1.2) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0,$

b) $C^i_{jk} y^j = C^i_{kj}\ y^j = 0\,,$

and c) $C^i_{jk}\, y_i = 0\,.$

Also satisfies the following relation:

(1.3) $C_{ijk} g^{jk} = C_i.$

The $(v)hv$- torsion tensor C^i_{jk} is the associate tensor of the tensor C_{ijk} is defined by

(1.4) a) $C^h_{ik} := g^{hj} C_{ijk}$ and b) $C_{ijk} := g_{hj} C^h_{ik}.$

The tensor C^h_{ik} is positively homogeneous of degree -1 in y^i and symmetric in its lower indices.

The tensor P^i_{jk} is called the $v(hv)$-torsion tensor and its given by

(1.5) $P^r_{jk} = (\dot\partial_j \Gamma^{*r}_{hk}) y^h = \Gamma^{*r}_{jhk}\ y^h\,.$

The tensor H^i_{jkh} satisfies the relation

(1.6) $H^{\,i}_{jkh}\, y^{\,j} = H^{\,i}_{kh}$.

The deviation tensor$H^{\,i}_{k}$ is positively homogeneous of degree two in $y^{\,i}$ and satisfies

(1.7) $H^{\,i}_{hk}\, y^{\,h} \;=\; H^{\,i}_{k}$.

The curvature tensor $K^{\,i}_{jkh}$ satisfies the following identities known as *Bianchi identities*

(1.8) $K^{\,r}_{ihk|j} + K^{\,r}_{ijh|k} + K^{\,r}_{ikj|h} + (\dot\partial_s \Gamma^{*r}_{ij}) K^{\,s}_{thk}\, y^{\,t} + (\dot\partial_s \Gamma^{*r}_{ik}) K^{\,s}_{tjh}\, y^{\,t} + (\dot\partial_s \Gamma^{*r}_{ih}) K^{\,s}_{tkj}\, y^{t} = 0$.

The associate tensor K_{ijkh} of the curvature tensor $K^{\,i}_{jkh}$ is given by

(1.9) $K_{ijkh} := g_{rj}\, K^{\,r}_{ikh}$.

The tensor K_{ijkh} also satisfies the condition

(1.10) $K_{jikh} + K_{ijkh} = -2\, C_{ijs}\, K^{\,s}_{rkh}\; y^{\,r}$.

The curvature tensor $K^{\,i}_{jkh}$ satisfies the following relations too

(1.11) $K^{\,i}_{jkh}\;\; y^{\,j} = H^{\,i}_{kh}$,

and (1.12) $H^{\,i}_{jkh} - K^{\,i}_{jkh} = P^{\,i}_{jk|h} + P^{\,r}_{jk}\, P^{\,i}_{rh} - h/k$,
where the tensor $H^{\,i}_{jkh}$ and $H^{\,i}_{kh}$form the Berwald Curvature tensor and the $h(v)$-torsion tensor respectively.

The Ricci tensor K_{jk} of the curvature tensor $K^{\,i}_{jkh}$ is given by

(1.13) $K^{\,i}_{jki} \;= K_{jk}$.

N. S. H. Hussein [4] introduced the K^h– recurrent space. Thus, the K^h– *recurrent space* characterized by

(1.14) $K^{i}_{jkh|\ell} = \lambda_\ell\, K^{i}_{jkh}$, $K^{i}_{jkh} \neq 0$,

where the non–zero covariant vector field λ_l being the recurrence vector field.

M. A. A. Ali [1] discussed the K^h– *birecurrent space*. Thus, the K^h– *birecurrent space* is characterized by

(1.15) $K^{i}_{jkh|\ell\,|m} = a_{\ell m}\, K^{i}_{jkh}$, $K^{i}_{jkh} \neq 0$,
where $a_{\ell m}$ is non–zero covariant tensor field of second order is called *the birecurrence tensor field*.

Differerntiating (1.14) covariently with respect to x^m in the sense of Cartan, we get

(1.16) $K^{i}_{jkh|\ell\,|m} = \lambda_\ell\, K^{i}_{jkh|m} + \lambda_{\ell|m}\, K^{i}_{jkh}$, $K^{i}_{jkh} \neq 0$,

which can be written as

(1.17) $K^{i}_{jkh|\ell\,|m} = \lambda_\ell\, K^{i}_{jkh|m} + b_{\ell m}\, K^{i}_{jkh}$, $K^{i}_{jkh} \neq 0$

where λ_ℓ and $b_{\ell m} = \lambda_{\ell|m}$ are non-zero covariant vector fields and covariant tensor field of second order, respectively. The space and the tensor satisfying the condition (1.17) will be called K^h*–generalized birecurrent space* and h*–generalized birecurrent tensor*, respectively. We shall denote them briefly by $K^h - GBR - F_n$ and $h - GBR$, respectively.

Transvecting (1.17) by the metric tensor g_{ip} , using (1.9) and (1.1c), we get

(1.18) $K_{jpkh|\ell|m} \;=\; \lambda_\ell K_{jpkh|m} + b_{\ell m}\, K_{jpkh}$.

Contracting the indices i and h in (1.17) and using (1.13), we get

(1.19) $K_{jk|\ell|m} = \lambda_\ell\, K_{jk|m} + b_{\ell m}\;\, K_{jk}$.

Transvecting (1.19) by y^k and using (1.1b), we get

(1.20) $K_{j|\ell|m} = \lambda_\ell\, K_{j|m} + b_{\ell m}\, K_j$.

where $K_{jk}\, y^k = K_j$.

Transvecting (1.17) by y^j , using (1.1b) and (1.11), we get

(1.21) $$H^i_{kh|\ell\,|m} = \lambda_\ell\, H^i_{kh|m} + b_{\ell m}\; H^i_{kh}\,.$$

Contracting the indices i and h in (1.21) and using ($H_k = H^i_{ki}$), we get

(1.22) $$H_{k|\ell|m} = \lambda_\ell\; H_{k|m} + b_{\ell m}\;\; H_k.$$

2. CERTAIN IDENTITIES

In view of the identity (1.10) and (1.11), we have

(2.1) $$K_{hijk} + K_{ihjk} = -2C_{hir}H^r_{jk}$$

Differentiating (2.1) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.2) $$K_{hijk|\ell} + K_{ihjk|\ell} = (-2C_{hir}\; H^r_{jk}\,)_{|\ell}$$

Differentiating (2.2) covariantly with respect to x^m in the sense of Cartan, we get

(2.3) $$K_{hijk|\ell|m} + K_{ihjk|\ell|m} = (-2C_{hir}\; H^r_{jk}\,)_{|\ell|m}$$

Using (1.18) in (2.3), we get

(2.4) $$\lambda_\ell(\, K_{hijk|m} + K_{ihjk|m}) + b_{\ell m}\big(K_{hijk} + K_{ihjk}\big) = (-2C_{hir}\; H^r_{jk}\,)_{|\ell|m}$$

Putting (2.1) and (2.2) in (2.4), we get

(2.5) $$(C_{hir}\; H^r_{jk}\,)_{|\ell|m} \;=\; \lambda_\ell(C_{hir}\; H^r_{jk}\,)_{|m} + b_{\ell m}\,(C_{hir}\; H^r_{jk}\,)$$

Transvecting (2.5) by g^{hp}, using (1.1d) and (1.4a), we get

(2.6) $$(C^p_{ir}H^r_{jk}\,)_{|\ell|m} \;=\; \lambda_\ell(C^p_{ir}H^r_{jk}\,)_{|m} + b_{\ell m}\,(C^p_{ir}H^r_{jk}\,)$$

Transvecting (2.6) by y^j, using (1.1b) and (1.7), we get

(2.7) $$(C^p_{ir}\; H^r_k\,)_{|\ell|m} \;=\; \lambda_\ell(C^p_{ir}\;\; H^r_k\,)_{|m} + b_{\ell m}\,(C^p_{ir}\;\; H^r_k\,)$$

Transvecting (2.5) by g^{hi}, using (1.1d) and (1.3), we get

(2.8) $$(C_r\; H^r_{jk}\,)_{|\ell|m} \;=\; \lambda_\ell(C_r\; H^r_{jk}\,)_{|m} + b_{\ell m}\,(C_r\; H^r_{jk}\,)$$

Transvecting (2.8) by y^j, using (1.1b) and (1.7), we get

(2.9) $$(C_r\; H^r_k\,)_{|\ell|m} \;=\; \lambda_\ell(C_r\; H^r_k\,)_{|m} + b_{\ell m}\,(C_r\; H^r_k\,)$$

Contracting the indices p and k in (2.7), we get

(2.10) $$(C^p_{ir}\; H^r_p\,)_{|\ell|m} \;=\; \lambda_\ell(C^p_{ir}\; H^r_p\,)_{|m} + b_{\ell m}\,(C^p_{ir}\; H^r_p\,)$$

Thus, we conclude

Theorem 2.1. *In* $K^{\,h}\!-GBR\!-F_n$, the tensors $(C_{hir}\;H^r_{jk}\,)$, $(C^p_{ir}H^r_{jk}\,)$, $(C_r\;H^r_{jk}\,)$, $(C_r\;H^r_k\,)$ and $(C^p_{ir}\;\;H^r_p\,)$ are all $h\!-GBR$.

We know the identity [6]

(2.11) $$K_j \;=\; H_j \;-\; H^i_j\, C_i$$

Differentiating (2.11) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.12) $$K_{j|\ell} \;=\; H_{j|\ell} \;-\; (\,H^i_j\, C_i)_{|\ell}$$

Differentiating (2.12) covariantly with respect to x^m in the sense of Cartan, we get

(2.13) $$K_{j|\ell|m} \;=\; H_{j|\ell|m} - (\,H^i_j\, C_i)_{|\ell|m}$$

Using (1.22) and (2.9) in (2.13), we get

(2.14) $K_{j|\ell|m} = \lambda_\ell \,(H_{j|m} - (H^i_j\, C_i)_{|m}) + b_{\ell m}\,(H_j - (H^i_j\, C_i)$

Putting (2.11) and (2.12) in (2.14), we get

(2.15) $K_{j|\ell|m} = \lambda_\ell\, K_{j|m} + b_{\ell m}\, K_j$

Thus, we conclude

Theorem 2.2. *In* $K^h\!-GBR\!-F_n$, the vector K_j is $h\!-GBR$.

Also, we have the identity [6]

(2.16) $R_j = K_j + C^i_{jr}\, H^r_i$

Differentiating (2.16) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.17) $R_{j|\ell} = K_{j|\ell} + (C^i_{jr}\, H^r_i)_{|\ell}$

Differentiating (2.17) covariantly with respect to x^m in the sense of Cartan, we get

(2.18) $R_{j|\ell|m} = K_{j|\ell|m} + (C^i_{jr}\, H^r_i)_{|\ell|m}$

Using (2.10) and (2.15) in (2.18), we get

(2.19) $R_{j|\ell|m} = \lambda_\ell\,(K_{j|m} + (C^i_{jr}\, H^r_i)_{|m}) + b_{\ell m}(K_j + (C^i_{jr}\, H^r_i\,)$

Putting (2.16) and (2.17) in (2.19), we get

(2.20) $R_{j|\ell|m} = \lambda_l\, R_{j|m} + b_{\ell m} R_j$

Thus, we conclude

Theorem 2.3. *In* $K^{\,h}\!-GBR\!-F_n$, the vector R_j is $h\!-GBR$.

We have the Cartan's fourth curvature tensor K^i_{jkh}, $v(\,hv\,)-$ torsion tensor P^i_{jk} and the Berwald curvature tensor H^i_{jkh} are connected by the formula (1.12)

$H^{\,i}_{jkh} - K^{\,i}_{jkh} = P^i_{jk|h} + P^r_{jk}\, P^i_{rh} - h/k^*.$

Differentiating (1.12) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.21) $H^{\,i}_{jkh|\ell} - K^{\,i}_{jkh|\ell} = (P^i_{jk|h} + P^r_{jk}\, P^i_{rh} - h/k^*)_{|\ell}$

Differentiating (2.21) covariantly with respect to x^m in the sense of Cartan, we get

(2.22) $H^{\,i}_{jkh|\ell|m} - K^{\,i}_{jkh|\ell|m} = (P^i_{jk|h} + P^r_{jk}\, P^i_{rh} - h/k^*)_{|\ell|m}$

Using (1.17) and if Berwald curvature tensor H^i_{jkh} is $h\!-GBR$, (2.22) reduce to

(2.23) $\lambda_\ell\big(H^{\,i}_{jkh|m} - K^{\,i}_{jkh|m}\big) + b_{\ell m}\big(H^{\,i}_{jkh} - K^{\,i}_{jkh}\big) = (P^i_{jk|h} + P^r_{jk}\, P^i_{rh} - h/k^*)_{|\ell|m}$

Putting (1.12) and (2.21) in (2.23), we get

(2.24) $(P^i_{jk|h} + P^r_{jk}P^i_{rh} - h/k)_{|\ell|m} = \lambda_\ell(P^i_{jk|h} + P^r_{jk}P^i_{rh} - h/k)_{|m} + b_{\ell m}\big(P^i_{jk|h} + P^r_{jk}P^i_{rh} - h/k\big)$

Thus, we conclude

Theorem 2.4. *In* $K^h\!-GBR\!-F_n$, the tensor $\big(P^i_{jk|h} + P^r_{jk}P^i_{rh} - h/k\big)$ is $h\!-GBR$, [provided $v(\,hv\,)-$ torsion tensor P^i_{jk} and Berwald curvature tensor H^i_{jkh} are all $h\!-GBR$].

We know the curvature tensor K_{ijkh} satisfies[5],[8] the identity

(2.25) $K_{hijk} - K_{jkhi} = H^r_{hj} C_{rik} - H^r_{hk}\, C_{rij} + H^r_{ik} C_{rhj} - H^r_{ij}\ C_{rhk} - H^r_{jk}\, C_{rhi} + H^r_{hi}\ C_{rjk}.$

Differentiating (2.25) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.26) $K_{hijk|\ell} - K_{jkhi|\ell} = (H^r_{hj} C_{rik} - H^r_{hk} C_{rij} + H^r_{ik} C_{rhj} - H^r_{ij} C_{rhk} - H^r_{jk} C_{rhi} + H^r_{hi} C_{rjk})_{|\ell}$

Differentiating (2.26) covariantly with respect to x^m in the sense of Cartan, we get

(2.27) $K_{hijk|\ell|m} - K_{jkhi|\ell|m} = (H^r_{hj} C_{rik} - H^r_{hk} C_{rij} + H^r_{ik} C_{rhj} - H^r_{ij} C_{rhk} - H^r_{jk} C_{rhi} + H^r_{hi} C_{rjk})_{|\ell|m}$

Using (1.18) in (2.27), we get

(2.28) $\lambda_\ell(K_{hijk|\ell} - K_{jkhi|\ell}) + b_{\ell m}(K_{hijk} - K_{jkhi}) = (H^r_{hj}\ C_{rik} +$

$b_{\ell m}\left(C_{iks} H^s_h - C_{ihs} H^s_k\right) H^r_{hj}\ C_{rik} + H^r_{hj}\ C_{rik} + b_{\ell m}\left(C_{iks} H^s_h - C_{ihs} H^s_k\right)$

$H^r_{hj}\ C_{rik} + b_{\ell m}\left(C_{iks} H^s_h - C_{ihs} H^s_k\right) H^r_{hj}\ C_{rik} + H^r_{hj}\ C_{rik})_{|\ell|m}$

Putting (2.25) and (2.26) in (2.28), we get

(2.29) $(H^r_{hj} C_{rik} - H^r_{hk} C_{rij} + H^r_{ik} C_{rhj} - H^r_{ij}\ C_{rhk} - H^r_{jk} C_{rhi} + H^r_{hi} C_{rjk})_{|\ell|m}$

$= \lambda_\ell(H^r_{hj} C_{rik} - H^r_{hk} C_{rij} + H^r_{ik} C_{rhj} - H^r_{ij} C_{rhk} - H^r_{jk}\ C_{rhi} + H^r_{hi}\ C_{rjk})_{|m}$

$+ b_{\ell m}\left(H^r_{hj} C_{rik} - H^r_{hk} C_{rij} + H^r_{ik} C_{rhj} - H^r_{ij}\ C_{rhk} - H^r_{jk} C_{rhi} + H^r_{hi} C_{rjk}\right)$

Transvecting (2.29) by y^j , using (1.1b), (1.2a) and (1.7), we get

(2.30) $(H^r_h C_{rik} - H^r_i C_{rhk} + H^r_k C_{rhi})_{|\ell|m} = \lambda_\ell(H^r_h C_{rik} - H^r_i C_{rhk} + H^r_k C_{rhi})_{|m}$

$+ b_{\ell m}\ (H^r_h C_{rik} - H^r_i C_{rhk} + H^r_k C_{rhi})$

Transvecting (2.30) by g^{pr} , using (1.1d) and (1.4a), we get

(2.31) $(H^r_h C^p_{ik} - H^r_i C^p_{hk} + H^r_k C^p_{hi})_{|l|m} = \lambda_l(H^r_h C^p_{ik} - H^r_i C^p_{hk} + H^r_k C^p_{hi})_{|m} +$

$b_{\ell m}\ (H^r_h C^p_{ik} - H^r_i C^p_{hk} + H^r_k C^p_{hi})$

Thus, we conclude

Theorem 2.5.

In $K^h - GBR - F_n$, the tensors $\left(H^r_{hj} C_{rik} - H^r_{hk} C_{rij} + H^r_{ik} C_{rhj} - H^r_{ij}\ C_{rhk} - H^r_{jk}\ C_{rhi} + H^r_{hi}\ C_{rjk}\right)$,$(H^r_h C_{rik} - H^r_i C_{rhk} + H^r_k C_{rhi})$ and $(H^r_h C^p_{ik} - H^r_i C^p_{hk} + H^r_k C^p_{hi})$ are all $h - GBR$.

We have the identity [6]

(2.32) $K_{ijhk} + K_{ikjh} + K_{ihkj} = -2\, y^r\left(C_{ijs} K^s_{rhk} + C_{iks} K^s_{rjh} + C_{ihs} K^s_{rkj} \right)$

Using (1.11) in (2.32), we get

(2.33) $K_{ijhk} + K_{ikjh} + K_{ihkj} = -2\left(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj} \right)$

Differentiating (2.33) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.34) $K_{ijhk|\ell} + K_{ikjh|\ell} + K_{ihkj|\ell} = -2\left(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj} \right)_{|\ell}$

Differentiating (2.34) covariantly with respect to x^m in the sense of Cartan, we get

(2.35) $K_{ijhk|\ell|m} + K_{ikjh|\ell|m} + K_{ihkj|\ell|m} = -2\left(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj} \right)_{|\ell|m}$

Using (1.18), (2.33) and (2.34) in (2.35), we get

(2.36) $\left(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj} \right)_{|\ell|m} = \lambda_\ell\left(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj} \right)_{|m}$

$+ b_{\ell m}\left(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj} \right)$

Transvecting (2.36) by y^j, using (1.1b), (1.2a) and (1.7), we get

(2.37) $(C_{iks}\, H^s_h - C_{ihs}\, H^s_k)_{|\ell|m} = \lambda_\ell (C_{iks}\, H^s_h - C_{ihs}\, H^s_k)_{|m} + b_{\ell m}\,(C_{iks}\, H^s_h - C_{ihs}\, H^s_k)$

Transvecting (2.37) by g^{pr}, using (1.1d) and (1.4a), we get

(2.38) $(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)_{|\ell|m} = \lambda_\ell (C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)_{|m} + b_{\ell m}\,(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)$

Thus, we conclude

Theorem 2.6. *In* $K^h\text{–}GBR\text{–}F_n$, the tensors $(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj})$, $(C_{iks}\, H^s_h - C_{ihs}\, H^s_k)$ and $(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)$ are all $h\text{–}GBR$.

The Bianchi identity for Cartan's fourth curvature tensor K^i_{jkh} is given by (1.8)

$$K^i_{jkh|\ell} + K^i_{j\ell k|h} + K^i_{jh\ell|k} + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})K^s_{rkh} + (\dot\partial_s \Gamma^{*i}_{jh})K^s_{r\ell k}\} = 0$$

Differentiating (1.8) covariantly with respect to x^m in the sense of Cartan, we get

(2.39) $$K^i_{jkh|\ell|m} + K^i_{j\ell k|h|m} + K^i_{jh\ell|k|m} + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})K^s_{rh\ell|m} + (\dot\partial_s \Gamma^{*i}_{j\ell})K^s_{rkh|m} + (\dot\partial_s \Gamma^{*i}_{jh})K^s_{r\ell k|m}\} + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} + (\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh|m} + K^s_{rh\ell|m}\,(\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k|m}\} = 0$$

Using (1.1b) and (1.17) in (2.39), we get

(2.40) $$\lambda_\ell K^i_{jkh|m} + \lambda_h K^i_{j\ell k|m} + \lambda_k K^i_{jh\ell|m} + b_{\ell m} K^i_{jkh} + b_{\ell h} K^i_{j\ell k} + b_{\ell k} K^i_{jh\ell} + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})K^s_{rh\ell|m} + (\dot\partial_s \Gamma^{*i}_{j\ell})K^s_{rkh|m} + (\dot\partial_s \Gamma^{*i}_{jh})K^s_{r\ell k|m}\} + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh} + (\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k}\} = 0$$

If Cartan fourth curvature tensor K^i_{jkh} is h– recurrent which is given by (1.14) and in view of (2.40), we get

(2.41) $$\lambda_\ell K^i_{jkh|m} + \lambda_h K^i_{j\ell k|m} + \lambda_k K^i_{jh\ell|m} + b_{\ell m} K^i_{jkh} + b_{\ell h} K^i_{j\ell k} + b_{\ell k} K^i_{jh\ell} + \lambda_m\, y^r\{(\dot\partial_s \Gamma^{*i}_{jk})K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})K^s_{rkh} + (\dot\partial_s \Gamma^{*i}_{jh})K^s_{r\ell k}\} + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh}\,(\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k}\} = 0$$

Putting (1.8) in (2.41) and using (1.14), we get

(2.42) $$\lambda_\ell K^i_{jkh|m} + \lambda_h K^i_{j\ell k|m} + \lambda_k K^i_{jh\ell|m} + (b_{\ell m} - \lambda_\ell\,\lambda_m)\,K^i_{jkh} + (b_{\ell h} - \lambda_\ell \lambda_h) K^i_{j\ell k} + (b_{\ell k} - \lambda_\ell \lambda_k) K^i_{jh\ell} + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh}\,(\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k}\} = 0$$

which can be written as

(2.43) $$\lambda_\ell K^i_{jkh|m} + \lambda_h K^i_{j\ell k|m} + \lambda_k K^i_{jh\ell|m} + a_{\ell m} K^i_{jkh} + a_{\ell h} K^i_{j\ell k} + a_{\ell k} K^i_{jh\ell} + + y^r\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh} + (\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k}\} = 0$$

where $a_{\ell r} = (b_{\ell r} - \lambda_\ell \lambda_r)$ is non–zero covariant tensor field of second order.

Transvecting (2.43) by y^j, using (1.1b), (1.11) and (1.5), we get

(2.44) $$\lambda_\ell H^i_{kh|m} + \lambda_h H^i_{\ell k|m} + \lambda_k H^i_{h\ell|m} + a_{\ell m} H^i_{kh} + a_{\ell h} H^i_{\ell k} + a_{\ell k} H^i_{h\ell} + P^i_{sk} H^s_{h\ell} + P^i_{s\ell} H^s_{kh} + P^i_{sh} H^s_{\ell k} = 0$$

Thus, we conclude

Theorem 2.7. *In* $K^h - GBR - F_n$, we have the identities (2.43) and (2.44) [provided Cartan fourth curvature tensor K^i_{jkh} is $h-$ recurrent].

We know that the associate tensor R_{ijkh} of Cartan third curvature tensor R^i_{jkh} satisfies the identity[2],[3]

(2.45) $R_{ijhk} + R_{ikjh} + R_{ihkj} + (C_{ijs}\, K^s_{rhk} + C_{iks}\, K^s_{rjh} + C_{ihs}\, K^s_{rkj})y^r = 0$

Using (1.11) in (2.45), we get

(2.46) $R_{ijhk} + R_{ikjh} + R_{ihkj} + C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj} = 0$

Differentiating (2.46) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.47) $R_{ijhk|\ell} + R_{ikjh|\ell} + R_{ihkj|\ell} + (C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj})_{|\ell} = 0$

Differentiating (2.47) covariantly with respect to x^m in the sense of Cartan and if the associate tensor R_{ijkh} of Cartan third curvature tensor R^i_{jkh} is $h-GBR$, we get

(2.48) $\lambda_\ell(R_{ijhk|m} + R_{ikjh|m} + R_{ihkj|m}) + b_{\ell m}(R_{ijhk} + R_{ikjh} + R_{ihkj})$

$+(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj})_{|\ell|m} = 0$

In view of (2.47) and putting (2.46) in (2.48), we get

(2.49) $(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj})_{|\ell|m} = \lambda_\ell(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj})_{|m}$

$+ b_{\ell m}(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj})$

Transvecting (2.49) by y^j, using (1.1b), (1.2a) and (1.7), we get

(2.50) $(C_{iks}\, H^s_h - C_{ihs}\, H^s_k)_{|l|m} = \lambda_\ell(C_{iks}\, H^s_h - C_{ihs}\, H^s_k)_{|m} + b_{\ell m}(C_{iks}\, H^s_h - C_{ihs}\, H^s_k)$.

Transvecting (2.50) by g^{pi}, using (1.1d) and (1.4a), we get

(2.51) $(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)_{|\ell|m} = \lambda_\ell(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)_{|m} + b_{\ell m}(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)$

Thus, we conclude

Theorem 2.8. *In* $K^h - GBR - F_n$, the tensors $(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj})$, $(C_{iks}\, H^s_h - C_{ihs}\, H^s_k)$ and $(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k)$ are all $h-GBR$ [provided the associate tensor R_{ijkh} of Cartan third curvature tensor R^i_{jkh} is $h-GBR$].

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